Singular sources mining using evidential conflict analysis

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Abstract

Singular sources mining is essential in many applications like sensor fusion or dataset analysis. A singular source of information provides pieces of evidence that are significantly different from the majority of the other sources. In the Dempster-Shafer theory, the pieces of evidence collected by a source are summarized by basic belief assignments (bbas). In this article, we propose to mine singular sources by analysing the conflict between their corresponding bbas. By viewing the conflict as a function of parameters called discounting rates, new developments are obtained and a criterion that weights the contribution of each bba to the conflict is introduced. The efficiency and the robustness of this criterion is demonstrated on several sets of bbas with various specificities.

Keywords: Dempster-Shafer Theory, Conflict analysis, Outlier detection

1. Introduction

The Belief Function Theory (BFT), also known as the Dempster-Shafer theory [1, 2], has gained popularity because it can process data that are not only uncertain but also imprecise and then aggregate these different data using a combination rule. When tackling data fusion problem, a major difficulty to resolve is how to deal with conflicting pieces of information. The BFT allows the computation of a measure called the degree of conflict. This measure is an indication on how much the sources of information, from which data are originated, are in conflict. Smets [3] analysed combination processes in case of conflicting sources *i.e.* when the degree of conflict is positive. Yet the degree of conflict has a major drawback in the sense that it does not evaluate how each source individually contributes to the conflict.

A more refined analysis of conflicting pieces of evidence can indeed bring valuable information as it allows to outline that some data appears to be singular as compared to the whole data collection. Singularity ranges from a situation

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where a piece of evidence is completely isolated to a situation where a piece of evidence shares common view with a substantial proportion of the collected pieces. It is thus a notion that needs to be gradually evaluated in order to be efficiently integrated inside an information processing system.

Individual evaluations of the singularity of pieces of evidence are notably of great importance in the field of outlier, fault or novelty detection. Hodge and Austin [4] propose an extensive survey of outlier detection methodologies. The safety performances at stake are presented and a broad range of approaches are analysed among which statistical methods, neural networks and machine learning are found. There are ties between conflicting and outlying data and we believe that the degree of conflict encompasses precious information toward the identification of singular or outlying data, hence the motivation to investigate on new conflict analysis criteria.

Martin *et al.* [5] and Schubert [6, 7] have both proposed criteria that allow an individual measure of conflict for each bba involved into a combination process. However both of these approaches are highly dependent on the proportion of singular bbas and the total number of processed bbas.

These dependencies make them difficult to use in contexts where these two quantities may vary. We propose in this article a new criterion that is more robust to the variations of the proportion of conflicting bbas as well as to the number of bbas. This criterion is derived by analysing the degree of conflict as a function of discounting rates. Discounting rates are used as part of a belief updating mechanism. A bba yielded by a source that is known to be unreliable is assigned a large discounting rate so that its weight in the combination process is reduced.

The first section of this paper presents general facts about belief functions and the BFT. The second section contains an overview of conflicting bba characterization methods. Our new criterion is presented and justified. The third section presents experiments on synthetic sets of bbas. These sets are made of various kinds of bbas so as to highlight the differences between the proposed criterion and existing criteria.

2. Dempster-Shafer Theory: fundamental concepts

2.1. Problem modelling

The BFT provides a formal framework for dealing with both imprecise and uncertain data. The finite set of mutually exclusive solutions is denoted by $\Omega = \{\omega_1, ..., \omega_K\}$ and is called the **frame of discernment**. The set of all subsets of Ω is denoted by 2^Ω. A source S_i collects pieces of evidence leading to the assignment of belief masses to some elements of 2^Ω. The mass of belief assigned to A by S_i is denoted $m_i(A)$. The function m_i is called **basic belief assignment** (**bba**) and is such that:

$$m_i : 2^\Omega \to [0,1] \tag{1}$$

$$\sum_{A \subseteq \Omega} m_i(A) = 1. \tag{2}$$

The set of all bbas is denoted by $\mathfrak{B}^{\Omega}$.
A set A such that $m_i(A) > 0$ is called a **focal element** of m_i. Two elements of 2^{Ω} represents hypotheses with noteworthy interpretations:

- $\emptyset$: the solution of the problem may not lie within Ω.
- Ω: the problem's solution lies in Ω but is undetermined.

Considering the interpretation of $\emptyset$, two assumptions can be made concerning the frame of discernment. The open-world assumption states that $m(\emptyset) > 0$ is possible. The closed-world assumption bans $\emptyset$ from any belief assignments.
A bba is denoted by A^x if it has two focal elements: Ω and $A \subsetneq \Omega$, and if:

$$A^x(A) = 1 - x \text{ and } A^w(\Omega) = x. \tag{3}$$

with $x \in [0,1]$. Such bbas are called **simple bbas** (sbbas). By extension of this notation, the bba denoted by A^0 stands for the certainty that the truth belongs to A. Thus, Ω^0 stands for total ignorance ($\Omega^0(\Omega) = 1$); it is called the vacuous bba. This set of bbas are called **categorical bbas**.
Furthermore, a bba such that $m(\Omega) = 0$ is said to be **dogmatic**. It is said to be **normalized** if $m(\emptyset) = 0$.

2.2. Pieces of evidence combination

Suppose one has obtained two bbas from two distinct[1] pieces of evidence Ev_1 and Ev_2 collected respectively by sources S_1 and S_2. Let us further imagine that Ev_2 states that the solution of the problem lies for sure in a set $A \subset \Omega$. This piece of information is thus represented by the bba $m_2 = A^0$.
The basic combination problem consists in finding a way to aggregate m_1 and A^0. In the bayesian framework, this situation is known as conditioning and it is named likewise in the BFT. To integrate the information represented by A^0 into m_1, an intuitive solution is to reassign any mass allocated to a focal element B of m_1 to the intersection of A and B. This leads to the following formula :

$$m_1[A](X) = \sum_{B | B \cap A = X} m_1(B) \tag{4}$$

with $m_1[A]$ the bba m_1 contionned on A. Now, one can generalize this process to define a combination rule between any bbas m_1 and m_2. This leads to the definition of the most commonly used combination rule in the BFT : the conjunctive rule $\textcircled{\cap}$:

$$\forall X \in 2^{\Omega}, m_{1 \textcircled{\cap} 2}(X) = \sum_{B,C | B \cap C = X} m_1(B) m_2(C). \tag{5}$$

[1]Pieces of evidence are distinct if the construction of beliefs according to one piece of evidence does not restrict the construction of beliefs using another piece of evidence.

This combination rule is compliant with the open-world assumption. The closed-world counterpart of the conjunctive rule is known as Dempster's combination rule $\oplus$:

$$\forall X \neq \emptyset, m_{1\oplus 2}(X) = \frac{1}{1-\kappa} \sum_{B,C|B\cap C=X} m_1(B)\, m_2(C), \tag{6}$$

$$\text{with } \kappa = \sum_{B,C|B\cap C=\emptyset} m_1(B)\, m_2(C). \tag{7}$$

The mass $\kappa = m(\emptyset)$ is a normalization factor. This mass is given support when S_1 and S_2 advocate respectively for non-intersecting solutions. It is thus an indication on how much the two sources disagree, that is why it is named the **degree of conflict**. For the convenience of developments to come, the degree of conflict resulting from the combination of a set $\mathcal{S} = (m_1, ..., m_M)$ of M bbas is denoted $\kappa_{\mathcal{S}}$

The bba $m_{1 \bigcirc\!\!\!\!\cap\, 2}$ resulting from the conjunctive combination mentioned above assigns larger weights to small sets of Ω. Consequently, the amount of uncertainty is reduced and $m_{1 \bigcirc\!\!\!\!\cap\, 2}$ is said to be more **committed** than both m_1 and m_2. More precisely, $m_{1 \bigcirc\!\!\!\!\cap\, 2}$ is a specialization of both m_1 and m_2. m_i is a specialization of m_j, if there exists a square matrix Spe with general term $Spe(A, B)$, $A, B \subseteq \Omega$ verifying:

$$m_i(A) = \sum_{B\subseteq\Omega} Spe(A, B)\, m_j(A), \ \forall A \subseteq \Omega \tag{8}$$

with Spe such that :

$$\sum_{B\subseteq\Omega} Spe(A, B) = 1, \ \forall A \subseteq \Omega \tag{9}$$

$$Spe(A, B) > 0 \Rightarrow A \subseteq B, \ \forall A, B \subseteq \Omega. \tag{10}$$

This leads to the definition of a partial order $m_i \sqsubseteq m_j$ on $\mathfrak{B}^\Omega$.
Note that the two rules presented in this subsection are associative and commutative, therefore the order with which a whole set of bbas are combined does not matter. In addition, many other combination rules can be defined, one may find overviews of combination rules in [8], [9].

2.3. The discounting operation

It is possible to reduce the impact of a source of information and its corresponding bba using an operation called discounting [2]. This can be required for several reasons notably if the source of information is known to be unreliable or to enclose perishable information. Discounting m_i with discount rate $\alpha \in [0, 1]$ is defined as:

$$m_i^\alpha(X) = \begin{cases} (1-\alpha)m_i(X) \text{ if } X \neq \Omega, \\ (1-\alpha)m_i(X) + \alpha \text{ if } X = \Omega. \end{cases} \tag{11}$$

The higher α is, the stronger the discounting. Thanks to discounting, a source's bba is transformed into a function closer to the vacuous bba and $m_i \sqsubseteq m_i^\alpha$. One may remark that a sbba A^x is A^0 discounted with rate x.
Mercier *et al.* [10] presented a refined discounting, in which discount rates are computed for each subset $X \subset \Omega$ and each bba. The discounting is consequently more precise and efficient. It is, however, necessary to have enough information allowing subset-specific computation.
Recent developments [11] [12] have led to the definition of more general bba correction mechanisms than the discounting operation. Instead of being discounted, a bba can also be notably re-inforced.
In this paper, our study is limited to classical discounting as defined in equation (11). As part of sequential approches [7] or iterative methods [13], it is sometimes needed to discount a bba m_i sequentially: $m_i^{\alpha_1 \circ \alpha_2} = \left(m_i^{\alpha_1}\right)^{\alpha_2}$ with $\circ$ the composition law for successive discountings. If discountings are repeated n times with rates $(\alpha_1, ..., \alpha_n)$, one has the following property [3] :

$$m_i^\beta = m_i^{\alpha_1 \circ \ldots \circ \alpha_n}, \tag{12}$$

$$\text{with } \beta = 1 - \prod_{i=0}^{n} (1 - \alpha_i)\,. \tag{13}$$

Note that when $\forall i, \alpha_i = \alpha$, we have $\beta = 1 - (1 - \alpha)^n$.

3. Singular bbas mining using the degree of conflict

3.1. Problem statement and related works

A classical tricky situation with which data fusion system developers have to deal with is to combine a set of sources among which some conflicting sources are found. As experienced by these developers, the conflict observed does not originate from conflicting sources in identical proportions. In particular, singular sources prevail in conflict generation. By singular, one may understand a source that delivers information that is significantly different from the rest of the sources. Being singular or not is thus dependent on the fact that a source is in accordance with the majority opinion or not.
The difficulty is that a source can either be singular or outlying:

- because the device or the agent from which information is derived are respectively faulty or unreliable,
- because this source has detected some evidence to which other sources are blind, and this source is consequently very informative.

In the first case, singular sources must be detected and treated as erroneous to make the fusion process more robust. In the second case, singular sources must be identified and trigger an *ad hoc* process so that the information it contains is not lost or under-weighted as compared to the majority opinion.
In the BFT, each source S_i yields a bba m_i and consequently, singular source

mining and singular bba mining are understood in the same way in the rest of the paper. BFT approaches for outlier detection are proposed in [14, 15] but they do not use BFT tools for the outlier detection itself. The BFT is used to aggregate pieces of evidence on the presence of an outlier or not. Instead, we propose here to investigate how the singularity of some evidence can be pinpointed by the BFT. Indeed, combining a set $\mathcal{S} = (m_1, ..., m_M)$ of M bbas derived from singular sources will always result in a positive value for κ. The greater the discordance is, and the larger the number of singular sources are, the higher κ will be. Some criteria related to the degree of conflict can consequently be defined so as to detect singular bbas but a more detailed analysis than a single valued measure is needed.

First, multiple bba-dependent values are needed in order to be able to evaluate the contribution of each bba to the conflict. Second, for each bba this value must represent its singularity as compared to the values assigned to the other bbas. There are many ways to obtain a criterion of this kind. Defining new criteria is a task that requires specifications before starting to design such a mathematical tool. As part of such specifications, formal properties need to be defined. Properties are a simple and clear way of stating the goal that we intend to reach.

As reasoning in the general case is not an easy task in the BFT, let us focus on a specific situation that is easy to interpret and analyse. Suppose one has to process the following bba set $\mathcal{S}_{spe}$: $\mathcal{S}_{spe} = s_1 \cup s_2$ and $\forall m \in s_1$, $m = A^x$, $\forall m \in s_2$, $m = B^x$ with $A \cap B = \emptyset$. Let us denote $M = |\mathcal{S}_{spe}|$ (with $|X|$ the cardinal of set X) and $r_i = \frac{|s_i|}{|\mathcal{S}|}$. r_1 represents the proportion of bbas belonging to s_1 and if $|s_1| < |s_2|$ these bbas are singular as compared to those belonging to s_2 which correspond to the majority group. All bbas are identically committed in this example therefore the value assigned to a bba should be a simple expression of r_1 or r_2. We thus propose the definition of the following homogeneity property:

Property 1. Let $\mathcal{S}_{spe}$ be a bba set defined as above and γ_i the value assigned to m_i and representing its contribution to the conflict. The criterion γ is homogeneous if when $m_i \in s_j$, we have $\gamma_i = h(r_j)$ with $h :]0, 1] \rightarrow [0, +\infty[$ a bijective decreasing function independent of M with $h(1) = 0$.

The function h has to be decreasing because a gradual evaluation of the singularity is needed: the larger the proportion of bbas belonging to a group is, the smaller its contribution to the conflict is. Note that when one has $r_1 = r_2 = 0.5$, then a homogeneous criterion assigns the same value to all bbas. It is also desirable for h to be bijective so that one can fully control the criterion on this simple specific situation: one proportion of singular bbas is associated to one value and conversely. Finally, h is also independent of M because the same proportion of singular bbas implies the same contribution to the conflict whether the bba set is large or not.

Recently, Schubert [6, 7] proposed a criterion c_i, called the degree of falsity, that identifies to some extent the contribution of each individual bba m_i involved in the computation of $\kappa_{\mathcal{S}}$:

$$c_i = \frac{\kappa_{\mathcal{S}} - \kappa_{\mathcal{S}\setminus\{m_i\}}}{1 - \kappa_{\mathcal{S}\setminus\{m_i\}}} \tag{14}$$

where $\mathcal{S} \setminus \{m_i\}$ is the set difference of $\mathcal{S}$ and $\{m_i\}$. It is clear that if m_i is the only bba advocating for a particular solution, there will be a huge drop from $\kappa_{\mathcal{S}}$ to $\kappa_{\mathcal{S}\setminus\{m_i\}}$. Consequently, this very singular bba will have a large degree of falsity.

This criterion can be addressed a criticism because if there are at least two singular bbas in $\mathcal{S}$, the drop will be far less large. Looking at the specific situation depicted by the set $\mathcal{S}_{spe}$, the degree of falsity is very efficient but when $r > 1/M$, the detection of singular bbas may be impaired. c_i is obviously non-linear with respect to r and these non-linearities are dependent on the total number M of bbas involved in the fusion process. The degree of falsity fails to possess the homogeneity property and cannot be fully controlled on such a simple set as $\mathcal{S}_{spe}$.

Another minor drawback of c_i is also met when categorical bbas are aggregated. One may well have $\kappa_{\mathcal{S}\setminus\{m_i\}} = 1$, meaning that removing m_i does not suffice to avoid the full conflict case. In this situation, the falsity criterion reaches an undetermined value. In this paper, we consider that this value is 0 since it is not possible to conclude on the falsity of m_i. It may be possible to introduce a parameter to prevent bbas from being categorical, but this implies an additional parameter tuning.

Martin *et al.* [5] have also introduced several criteria, called conflict measures, evaluating the conflict provoked by a bba as compared to a set of bbas. These criteria are defined using a distance d_{BPA} between bbas introduced by Jousselme *et al.* [16]:

$$d_{BPA}(m_1, m_2) = \sqrt{1/2 \left(\vec{m_1} - \vec{m_2}\right)^t D \left(\vec{m_1} - \vec{m_2}\right)} \tag{15}$$

with $\vec{m}$ a vector form[2] of the bba m and D a $2^N \times 2^N$ matrix whose elements are $D(A, B) = |A \cap B| \,/\, |A \cup B|$. Martin *et al.* propose then the following conflict measures $Conf_i$:

$$Conf_i = \frac{1}{M-1} \sum_{j=1, i \neq j}^{M} d_{BPA}(m_i, m_j) \tag{16}$$

$$\text{or } Conf_i = d_{BPA}(m_i, m_*) \tag{17}$$

with m_* the combination of bbas in $\mathcal{S} \setminus \{m_i\}$. m_* can be obtained using different combination rules or by using the mean. Furthermore, the authors propose to tune this measure using a function f :

$$f(Conf_i). \tag{18}$$

[2] Subsets of Ω can be indexed as A_i using a binary order and the vector form of m is just $\left(m(A_1), ..., m\left(A_{|2^\Omega|}\right)\right)$.

The heuristic choice for f indicated by the authors is $f(x) = 1 - \left(1 - x^{\lambda}\right)^{1/\lambda}$ and $\lambda = 1.5$
For some of the conflict measures, it can be shown that a bijective decreasing function h can be found when one processes a bba set such as $\mathcal{S}_{spe}$. However, these functions fail to be independent of M and consequently conflict measures are not homogeneous.
In this paper, we intend to introduce a new criterion that also evaluates the contribution to the conflict of each individual bba and that possesses the homogeneity property. Thanks to this property, the behaviour of this criterion would be more easily predictable and thus more robust and easy to adapt when used in real problems.

3.2. Analysing the degree of conflict as a function of discounting rates

The degree of conflict is a measure that indicates the intensity with which a set of bbas $\mathcal{S} = (m_1, ..., m_M)$ are in conflict. When this set is discounted using predefined rates $(\alpha_1, ..., \alpha_M)$ for each bba, one may wonder what is the impact on the degree of conflict, hence, the idea of analysing $\kappa_{\mathcal{S}}$ as a function of discounting rates :

$$\kappa_{\mathcal{S}}(\vec{\alpha}) = \left(\bigcirc\!\!\!\!\cap_{i=1}^{M} m_i^{\alpha_i}\right)(\emptyset) \tag{19}$$

with $\vec{\alpha} = (\alpha_1, ..., \alpha_M)$. Note that when brackets are omitted, we define $\kappa_{\mathcal{S}} = \kappa_{\mathcal{S}}\left(\vec{0}\right)$. Following this idea, new developments and interpretations can be derived. We present a few of them hereafter.
Using this representation, one of the first idea that comes to mind is to investigate partial derivatives of function $\kappa_{\mathcal{S}}$. This leads to proposition 1:

Proposition 1. $\forall \mathcal{S} = \{m_i\}_{i=1}^{M} \subset \mathfrak{B}^{\Omega}$, $|\mathcal{S}| = M > 1$, $\forall \vec{\alpha} \in [0,1]^M$,

$$\frac{\partial \kappa_{\mathcal{S}}}{\partial \alpha_i}(\vec{\alpha}) = \kappa_{\mathcal{S}\setminus\{m_i\}}\left(p_{\mathcal{S}\setminus\{m_i\}}(\vec{\alpha})\right) - \kappa_{\mathcal{S}}(\vec{\alpha} - \alpha_i \vec{e_i}) \tag{20}$$

with $(\vec{e}_i)_{i=1}^{M}$ the canonical basis of $\mathbb{R}^M$ and $p_s(\vec{\alpha})$, the projection of $\vec{\alpha}$ on the vectorial space generated by $(\vec{e}_i)_{i|m_i \in s}$.

Proof: see Appendix A

One first remark is that the derivatives are always negative because the calculation of $\kappa_{\mathcal{S}}(\vec{\alpha} - \alpha_i \vec{e_i})$ involves the same bbas as $\kappa_{\mathcal{S}\setminus\{m_i\}}\left(p_{\mathcal{S}\setminus\{m_i\}}(\vec{\alpha})\right)$ plus m_i and adding a bba to the combination can only increase the degree of conflict. This is also linked to the fact that $\kappa_{\mathcal{S}}$ decreases as one of the discounting rate increases.
In addition, a rather surprising result is that $\frac{\partial \kappa_{\mathcal{S}}}{\partial \alpha_i}(\vec{\alpha})$ is a constant function with respect to variable α_i and $\frac{\partial^2 \kappa_{\mathcal{S}}}{\partial \alpha_i^2}(\vec{\alpha}) = 0$.
Furthermore the most conflicting bbas among $\mathcal{S}$ at point $\vec{\alpha}$ yield the steepest slope. Indeed, the same remark as for the degree of falsity can be made : if m_i is

the only bba advocating for a particular solution, there will be a huge drop from $\kappa_{\mathcal{S}}(\vec{\alpha}-\alpha_i\vec{e_i})$ to $\kappa_{\mathcal{S}\setminus\{m_i\}}\left(p_{\mathcal{S}\setminus\{m_i\}}(\vec{\alpha})\right)$ and consequently the value of $\left|\frac{\partial\kappa_{\mathcal{S}}}{\partial\alpha_i}(\vec{\alpha})\right|$ will be high. Note that the numerator of the degree of falsity c_i is $\frac{\partial\kappa_{\mathcal{S}}}{\partial\alpha_i}\left(\vec{0}\right)$.
Looking at these remarks, $\frac{\partial\kappa_{\mathcal{S}}}{\partial\alpha_i}(\vec{\alpha})$ appears to be a relevant measure to assess a bba's contribution to the conflict. Yet, it suffers from the same criticisms as the degree of falsity measure and the conflict measures as it is not a homogeneous criterion. Notably, the response it provides on bba sets like $\mathcal{S}_{spe}$ is dependent on the number M of bbas.
Another question that may be raised when investigated the global conflict $\kappa_{\mathcal{S}}$ produced by a set $\mathcal{S}$ of M bbas is its links with other conflicts produced by subsets of $\mathcal{S}$. Indeed, if one intends to estimate the impact of m_i on the combination, it may be interesting to compute $\kappa_{\{m_i\}\cup s}$ with s a subset of bbas in conflict m_i.
In the rest of this article, we define as **sub-degree of conflict** or **sub-conflict** a quantity κ_s such that $s \subsetneq \mathcal{S}$. A mathematical link between the sub-degrees of conflict and the global degree of conflict is expressed through the following proposition :

Proposition 2. $\forall\mathcal{S}=\{m_i\}_{i=1}^{M}\subset\mathfrak{B}^{\Omega}$, $|\mathcal{S}|=M>1$, $\forall\vec{\alpha}\in[0,1]^M$,

$$\kappa_{\mathcal{S}}(\vec{\alpha})=\sum_{s\subseteq\mathcal{S},s\neq\emptyset}\kappa_s\prod_{i=1}^{M}f_s(\alpha_i) \tag{21}$$

with $f_s(\alpha_i)=\begin{cases}\alpha_i \text{ } \textit{if } m_i\notin s\\ 1-\alpha_i \text{ } \textit{if } m_i\in s\end{cases}$.

Proof: see Appendix A

At first sight, this proposition might not seem very interesting in the sense that the right term is a weighted sum of sub-degrees whose expression makes it hard to understand the meaning behind it. However, it is simply noteworthy that such a mathematical link between the sub-degrees of conflict and the global degree of conflict exists. More interestingly, one may apply this formula for a very specific vector of discounting rates : $\vec{\alpha}=\frac{1}{2}\vec{u}$, with $\vec{u}=(1,...,1)$. We thus derive the following corollary:

Corollary 1. $\forall\mathcal{S}\subset\mathfrak{B}^{\Omega}$, $|\mathcal{S}|=M>1$,

$$\kappa_{\mathcal{S}}\left(\frac{1}{2}\vec{u}\right)=\frac{1}{2^M}\sum_{s\subseteq\mathcal{S},s\neq\emptyset}\kappa_s \tag{22}$$

.

Proof: using proposition 2 with $\forall i,\alpha_i=\frac{1}{2}$, one gets $\forall s,\forall i,f_s(\alpha_i)=\frac{1}{2}$. The result is then immediately obtained.

This result shows that the sum of all sub-conflicts normalized by 2^M is equivalent to the global conflict when all bbas are discounted by $\frac{1}{2}$. If we differentiate equation (22), we deduce the following proposition:

Proposition 3. $\forall \mathcal{S} = \{m_i\}_{i=1}^{M} \subset \mathfrak{B}^{\Omega}$, $|\mathcal{S}| = M > 1$,

$$\frac{\partial \kappa_{\mathcal{S}}}{\partial \alpha_i}\left(\frac{1}{2}\vec{u}\right) = \frac{1}{2^{M-1}} \sum_{s \subseteq \mathcal{S}, s \neq \emptyset, m_i \in s} \left[\kappa_{s \setminus \{m_i\}} - \kappa_s\right]. \tag{23}$$

Proof: From corollary 1 we have $\forall \vec{\alpha} \in [0,1]^M$, $\kappa_{\mathcal{S}}\left(\frac{1}{2}\vec{u} \circ \vec{\alpha}\right) = \frac{1}{2^M} \sum_{s \subseteq \mathcal{S}, s \neq \emptyset} \kappa_s\left(\vec{\alpha}\right)$. Now by differentiating, we obtain:

$$\frac{\partial}{\partial \alpha_i} \kappa_{\mathcal{S}}\left(\frac{1}{2}\vec{u} \circ \vec{\alpha}\right) = \frac{1}{2^M} \sum_{s \subseteq \mathcal{S}, s \neq \emptyset} \frac{\partial}{\partial \alpha_i} \kappa_s\left(\vec{\alpha}\right) \tag{24}$$

Using proposition 1, we have

$$\frac{1}{2}\frac{\partial \kappa_{\mathcal{S}}}{\partial \alpha_i}\left(\frac{1}{2}\vec{u} \circ \vec{\alpha}\right) = \frac{1}{2^M} \sum_{s \subseteq \mathcal{S}, s \neq \emptyset, m_i \in s} \kappa_{s \setminus \{m_i\}}\left(p_{s \setminus \{m_i\}}\left(\vec{\alpha}\right)\right) - \kappa_s\left(\vec{\alpha} - \alpha_i \vec{e}_i\right) \tag{25}$$

If we use the equation above with $\vec{\alpha} = \vec{0}$, the proposition result is obtained.

This result appears to be more interesting to achieve our goal of determining how much a bba contributes to the conflict. Indeed $\left|\frac{\partial \kappa_{\mathcal{S}}}{\partial \alpha_i}\left(\frac{1}{2}\vec{u}\right)\right|$ is understood as **the average drop of sub-conflicts when removing m_i from the combination**. As compared to $\frac{\partial \kappa_{\mathcal{S}}}{\partial \alpha_i}\left(\vec{\alpha}\right)$, this criterion can better detect singular bbas when their number is large. Yet it is still a non-homogeneous criterion, hence the idea to further discount all bbas with $\frac{1}{2}$. Discounting n times by $\frac{1}{2}$ is equivalent to discount one time by $\left[1 - \left(\frac{1}{2}\right)^n\right]$, see equation (13). This leads to proposition 4:

Proposition 4. $\forall \mathcal{S} = \{m_i\}_{i=1}^{M} \subset \mathfrak{B}^{\Omega}$, $|\mathcal{S}| = M > 1$, $\forall n \in \mathbb{N}^*$

$$\kappa_{\mathcal{S}}\left(\left[1 - \left(\frac{1}{2}\right)^n\right]\vec{u}\right) = \sum_{s \subseteq \mathcal{S}, s \neq \emptyset} \gamma_n^M\left(|s|\right) \kappa_s \tag{26}$$

with $\gamma_n^M\left(|s|\right) = \frac{\left(2^n - 1\right)^{M-|s|}}{2^{nM}}$.

Proof: see Appendix A.

This result is a particular case of proposition 2 where the actual values of the weights can be expressed using $\gamma_n^M\left(|s|\right)$. Again, we may differentiate this result and obtain the following proposition:

Proposition 5. $\forall \mathcal{S} = \{m_i\}_{i=1}^M \subset \mathfrak{B}^\Omega$, $|\mathcal{S}| = M > 1$, $\forall n \in \mathbb{N}^*$

$$\frac{1}{2^n}\frac{\partial \kappa_{\mathcal{S}}}{\partial \alpha_i}\left(\left[1-\left(\frac{1}{2}\right)^n\right]\vec{u}\right) = \sum_{s\subseteq \mathcal{S}, m_i\in s} \gamma_n^M\left(|s|\right)\left[\kappa_{s\setminus\{m_i\}} - \kappa_s\right]. \tag{27}$$

.

Proof: Similar proof as proposition 3, using proposition 4 and 1.

As compared to proposition 3, it is now possible to obtain easily a weighted sum of drops of sub-conflicts when removing m_i from the combination. By examining these weights $\gamma_n^M(|s|)$, one may note that they are dependent on the cardinal of the subset of bbas whose sub-conflict is evaluated. If bbas are normalized, the most prominent weights are obtained for pairwise sub-conflicts, *i.e.* when $|s| = 2$. Indeed, we have

$$\frac{\gamma_n^M(2)}{\gamma_n^M(q)} = \left(2^n - 1\right)^{q-2}. \tag{28}$$

with $q > 2$ an integer number. So the smallest ratio is obtained for $q = 3$ and this ratio is $(2^n - 1)$, therefore when n is large, weights for sub-conflicts involving more than 2 bbas are negligible as compared to the weights for pairwise sub-conflicts. Given that when n is large $\gamma_n^M(2) \approx \frac{1}{4^n}$, we have for any set $\mathcal{S} = \{m_i\}_{i=1}^M$ of M normalized bbas such that $\exists m_j, m_k \in \mathcal{S}$ with $\kappa_{\{m_j,m_k\}} > 0$,

$$2^n\left|\frac{\partial \kappa_{\mathcal{S}}}{\partial \alpha_i}\left(\left[1-\left(\frac{1}{2}\right)^n\right]\vec{u}\right)\right| \approx \sum_{m_j\in\mathcal{S}\setminus\{m_i\}} \kappa_{\{m_i,m_j\}} \text{ when } n >> 1. \tag{29}$$

In practice, bba sets without any pairwise conflict but a positive global conflict are rarely found (see Appendix B for an example of such a situation). However, in this general case, given that when n is large $\gamma_n^M(q) \approx \frac{1}{2^{qn}}$, we have for any set $\mathcal{S} = \{m_i\}_{i=1}^M$ of M normalized bbas

$$2^{n(q-1)}\left|\frac{\partial \kappa_{\mathcal{S}}}{\partial \alpha_i}\left(\left[1-\left(\frac{1}{2}\right)^n\right]\vec{u}\right)\right| \approx \sum_{\substack{s\subseteq\mathcal{S}, m_i\in s\\ |s|=q}} \left[\kappa_{s\setminus\{m_i\}} - \kappa_s\right] \text{ when } n >> 1. \tag{30}$$

with $q = \min\left\{|s|, s\in\mathcal{S} \text{ such that } \kappa_s > 0\right\}$ the size of the smallest subset with a positive sub-conflict. We thus introduce criterion ξ_i:

$$\xi_i = \frac{1}{C_M^{q-1}} \sum_{s\subseteq\mathcal{S}, m_i\in s, |s|=q} \left[\kappa_{s\setminus\{m_i\}} - \kappa_s\right] \tag{31}$$

with C_M^{q-1} the binomial coefficient. When one processes a bba set such as $\mathcal{S}_{spe}$, we have $q = 2$. It can be shown that under such circumstances if $m_i \in s_j$ we obtain $\xi_i = r_j\left(1-x\right)^2$. So not only ξ is a homogeneous criterion, but its function h is linear with respect to the proportion of singular bbas.

3.3. Implementation details and parameter tuning for criterion ξ_i

This subsection gives an algorithmic solution to compute criterion ξ_i and discusses the influence of the parameters needed for its computation.
Indeed, two parameters arise from equation (30): n and q. Parameter n is related to the precision of the estimation of ξ_i, but the exact precision cannot be determined beforehand. Indeed, suppose the bba set is such that $q = 2$, the risk is that the chosen value of n is not enough to prevent a sub-conflict κ_s with $|s| = 3$ from polluting the estimation. To obtain a reliable estimation, n must be incremented until the absolute difference between two subsequent estimations of ξ_i becomes smaller than the desired precision ϵ. The initial value of n, denoted as n_{init}, should be such that $\frac{1}{2^n-1} << \epsilon$, so that only two successive computations are likely to be enough. In addition, n_{init} should also be chosen so that the machine precision is not reached and a bba $m_i^{1-\frac{1}{2}^n}$ turn into the vacuous bba.
Note that there remains a slight possibility that after convergence of the loop, the estimation obtained may be $\frac{1}{C_M^{p-1}} \sum_{s \in \mathcal{S}, m_i \in s, |s|=p} \left[\kappa_{s \setminus \{m_i\}} - \kappa_s\right]$ with $p > q$, but this would mean that sub-conflicts κ_s with $|s| < r$ are negligible as compared to sub-conflicts κ_s with $|s| = r$. Consequently, ξ_i remains a fair and relevant estimation of the conflict induced by bba m_i.
Concerning parameter q, it can be estimated easily using two subsequent estimations of ξ_i. Using equation (30), we can write:

$$1 \approx \frac{2^{(n+1)(q-1)} \left| \frac{\partial \kappa_{\mathcal{S}}}{\partial \alpha_i} \left(\left[1 - \left(\frac{1}{2}\right)^{n+1}\right] \vec{u} \right) \right|}{2^{n(q-1)} \left| \frac{\partial \kappa_{\mathcal{S}}}{\partial \alpha_i} \left(\left[1 - \left(\frac{1}{2}\right)^{n}\right] \vec{u} \right) \right|} \tag{32}$$

$$2^{q-1} \approx \frac{\left| \frac{\partial \kappa_{\mathcal{S}}}{\partial \alpha_i} \left(\left[1 - \left(\frac{1}{2}\right)^{n}\right] \vec{u} \right) \right|}{\left| \frac{\partial \kappa_{\mathcal{S}}}{\partial \alpha_i} \left(\left[1 - \left(\frac{1}{2}\right)^{n+1}\right] \vec{u} \right) \right|} \tag{33}$$

$$q \approx 1 + \log_2 \left(\frac{\left| \frac{\partial \kappa_{\mathcal{S}}}{\partial \alpha_i} \left(\left[1 - \left(\frac{1}{2}\right)^{n}\right] \vec{u} \right) \right|}{\left| \frac{\partial \kappa_{\mathcal{S}}}{\partial \alpha_i} \left(\left[1 - \left(\frac{1}{2}\right)^{n+1}\right] \vec{u} \right) \right|} \right) \tag{34}$$

with $\log_2$ the logarithm to base 2. The procedure to obtain criteria $\{\xi_i\}_{i=1}^M$ from a set $\mathcal{S}$ of M normalized bbas is given by the algorithm 1.

In addition to these comments, it is alo worth mentioning that criterion ξ_i can be used as an input of a function $g(r)$ corresponding to a specific desired behaviour with respect to r the proportion of singular bbas. Indeed since ξ_i is homogeneous and that its h function is linear, one can simply directly use $g(\xi_i)$ as an adapted criterion. In other words, one can easily derive a new criterion with any shape as a function of r. Examples of possible g functions are evoked in the experiments presented in the following section.

Algorithm 1 Computation of criterion ξ_i

entries : n_{init}, ϵ, $\mathcal{S}$, M
$n \leftarrow n_{init}$
repeat
  for i=1 to M **do**
    Compute $K_i^n \leftarrow \frac{\partial \kappa_{\mathcal{S}}}{\partial \alpha_i}\left(\left[1-\left(\frac{1}{2}\right)^n\right]\vec{u}\right)$ using equation (20)
    Compute $K_i^{n+1} \leftarrow \frac{\partial \kappa_{\mathcal{S}}}{\partial \alpha_i}\left(\left[1-\left(\frac{1}{2}\right)^{n+1}\right]\vec{u}\right)$ using equation (20)
    Compute q using equation (34)
    $n \leftarrow n+1$
  end for
until $\max_i \left|2^{n(q-1)}K_i^n - 2^{(n+1)(q-1)}K_i^{n+1}\right| < \epsilon$
for i=1 to M **do**
  $\xi_i \leftarrow \frac{2^{n(q-1)}}{M} K_i^n$
end for
return $\{\xi_i\}_{i=1}^{M}$ and q.
End

4. Experiments on synthetic sets of bbas

In this section several criteria evaluating the contribution to the conflict of bbas are compared using synthetic sets of bbas. We compare :

- the degree of falsity c_i,
- the conflict measure $Conf_i$ with $m*$ the mean of bbas ($m*$ is involved in the computation of $Conf_i$, see equation (17)),
- the criterion ξ_i with $n_{init} = 20$ and $\epsilon = 0.001$.

The two first experiments are meant to outline that ξ_i is homogeneous and why this property makes it easier to use than other criteria. The third experiment evaluate the performances of the 3 examined criteria in terms of conflict contribution evaluation. The fourth experiment describe the behaviour of the 3 examined criteria in a more general context and finally the last subsection presents a use case of the criteria.

4.1. Sets of sbbas with a varying proportion of singular bbas

In this experiment, we use the three criteria for the set $\mathcal{S}_{spe}$ presented in subsection 3.1. This set is the union of two subsets s_1 and s_2 that are respectively made of bbas A^x and B^x with $A \cap B = \emptyset$. We choose $M = 20$ bbas. For a given value of x, all bbas in $\mathcal{S}_{spe}$ are identically committed. Figure 1 shows the linearity of criterion ξ_i with respect to $r_1 = \frac{|s_1|}{M}$ for several values of x. The same curves are obtained for the set s_2. Figure 2 shows the non-linearity of criterion $Conf_i$ with respect to r_1 for several values of x. The same curves are obtained for the set s_2. It can be shown that $Conf_i^2$ is linear with respect to r_1. Consequently, if one chooses $g(r) = \sqrt{r}$, $g(\xi_i)$ is an adapted criterion whose behaviour is close to $Conf_i$. Figure 3 shows the non-linearity of criterion c_i

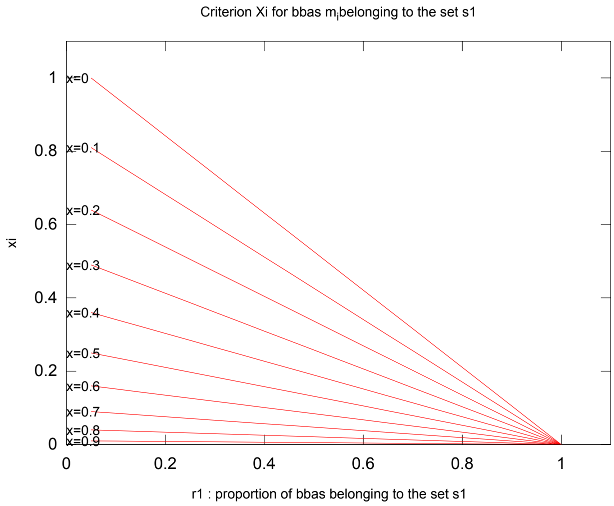

Figure 1: Behaviour of criterion ξ_i when the proportion of singular bbas varies.

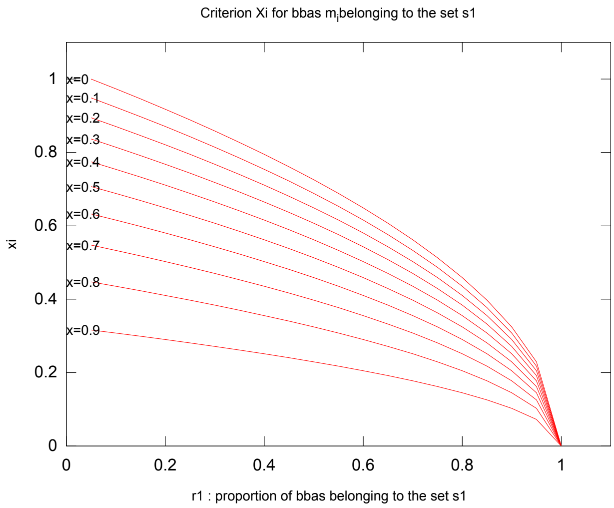

Figure 2: Behaviour of criterion $Conf_i$ when the proportion of singular bbas varies.

with respect to r_1 for several values of x. The same curves are obtained for the set s_2. Regarding this experiment, criterion c_i is the one whose behaviour is the most difficult to predict because an expression relating r_1 to c_i is hard to obtain. The value it takes appears to be rather binary depending on the fact that $|s_1| < |s_2|$ or $|s_1| > |s_2|$. Note that if one chooses a sigmoid function for $g(r)$, $g(\xi_i)$ is an adapted criterion whose behaviour is likely to be close to c_i. In addition, as explained in subsection 3.1, the criterion c_i fails when $x = 0$ *i.e.* when bbas are categorical.

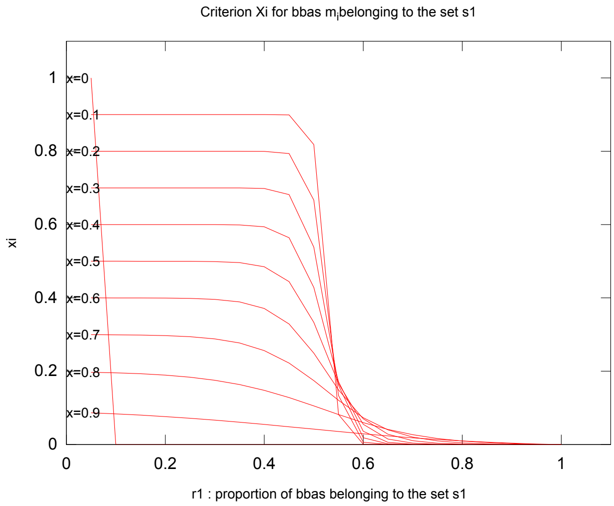

Figure 3: Behaviour of criterion c_i when the proportion of singular bbas varies.

4.2. Sets of bbas with varying number of bbas

One of the most interesting aspect of the homogeneity property is that on simple bba sets like $\mathcal{S}_{spe}$ the criterion value does not depend on M but only on the proportion of singular bbas. This is a very important property when the number of bbas varies with time like in the case of *ad hoc* networks or dynamic sensor networks. Both the conflict measure and the degree of falsity fail to possess such a property as it can be seen in Table 1.

Note that the dependency of $Conf_i$ decreases as M increases. Again, there is a computational limit for c_i when M is large.

4.3. Sets of bbas with random masses

The two aspects highlighted in the previous subsections are only valid for a particular kind of bba sets. In broader cases, these properties are no longer valid, however, it can be expected from criterion ξ_i to maintain a satisfying behaviour in the general case thanks to the homogeneity property. To compare the three criteria on a more general basis, random sets of sbbas were generated. A randomly chosen sbba $m_i = A^x$ is obtained as follows:

- a focal set A is randomly chosen in $\left\{2^\Omega \setminus \Omega, \emptyset\right\}$ (with equal probability for each subset),
- the mass assigned to this set is $1-x$ with x randomly chosen in $[0,1]$ using a uniform distribution.

We first present a worked out example on a particular sbba set $\mathcal{S}$ with $M = 20$ in Table 2. We note that this bba set contains a majority of bbas in favour of hypothesis b. This is pointed out by the bba combination using the conjunctive

Table 1: Values of c_i, $Conf_i$ and ξ_i for several bba sets of type $\mathcal{S}_{spe}$ with a varying size M. The proportion of singular bbas is $r = 0.25$. All bbas are identically committed and $|\Omega| = 3$. – means that the criterion could not be computed because the machine precision was reached.

bba set of type $\mathcal{S}_{spe}$ with $\lvert\mathcal{S}_{spe}\rvert = M$	bba type	degree of falsity c_i	conflict measure $Conf_i$	criterion ξ_i
M=4	bba $m_i \in s_1$, $s1 = \left\{1 \times m^{0.1}_{\{a\}}\right\}$	0.79	0.89	0.48
	bba $m_i \in s_2$, $s1 = \left\{3 \times m^{0.1}_{\{b\}}\right\}$	0.11	0.52	0.16
M=8	bba $m_i \in s_1$, $s1 = \left\{2 \times m^{0.1}_{\{a\}}\right\}$	0.80	0.83	0.48
	bba $m_i \in s_2$, $s1 = \left\{6 \times m^{0.1}_{\{b\}}\right\}$	6.10e-3	0.48	0.16
M=16	bba $m_i \in s_1$, $s1 = \left\{4 \times m^{0.1}_{\{a\}}\right\}$	0.80	0.80	0.48
	bba $m_i \in s_2$, $s1 = \left\{12 \times m^{0.1}_{\{b\}}\right\}$	1.00e-5	0.46	0.16
M=32	bba $m_i \in s_1$, $s1 = \left\{8 \times m^{0.1}_{\{a\}}\right\}$	0.80	0.79	0.48
	bba $m_i \in s_2$, $s1 = \left\{24 \times m^{0.1}_{\{b\}}\right\}$	≈0	0.45	0.16
M=64	bba $m_i \in s_1$, $s1 = \left\{16 \times m^{0.1}_{\{a\}}\right\}$	–	0.78	0.48
	bba $m_i \in s_2$, $s1 = \left\{48 \times m^{0.1}_{\{b\}}\right\}$	–	0.45	0.16
M=128	bba $m_i \in s_1$, $s1 = \left\{32 \times m^{0.1}_{\{a\}}\right\}$	–	0.78	0.48
	bba $m_i \in s_2$, $s1 = \left\{96 \times m^{0.1}_{\{b\}}\right\}$	–	0.45	0.16
M=256	bba $m_i \in s_1$, $s1 = \left\{64 \times m^{0.1}_{\{a\}}\right\}$	–	0.78	0.48
	bba $m_i \in s_2$, $s1 = \left\{192 \times m^{0.1}_{\{b\}}\right\}$	–	0.45	0.16
M=512	bba $m_i \in s_1$, $s1 = \left\{128 \times m^{0.1}_{\{a\}}\right\}$	–	0.78	0.48
	bba $m_i \in s_2$, $s1 = \left\{384 \times m^{0.1}_{\{b\}}\right\}$	–	0.45	0.16
M=1024	bba $m_i \in s_1$, $s1 = \left\{256 \times m^{0.1}_{\{a\}}\right\}$	–	0.77	0.48
	bba $m_i \in s_2$, $s1 = \left\{768 \times m^{0.1}_{\{b\}}\right\}$	–	0.45	0.16

Table 2: Results of criterion ξ_i, degree of falsity and conflict measure on a particular set $\mathcal{S} = \{m_1, ..., m_{20}\}$ of randomly chosen sbbas with $|\Omega| = 3$.

sbba	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{c\}$	$\{a,c\}$	$\{b,c\}$	$\Omega = \{a,b,c\}$	c_i	$Conf_i$	ξ_i
m_1	0	0.67	0	0	0	0	0	0.33	0.62	0.69	0.22
m_2	0	0	0.53	0	0	0	0	0.47	0.17	0.50	0.12
m_3	0	0	0.59	0	0	0	0	0.41	0.21	0.54	0.13
m_4	0	0	0.6	0	0	0	0	0.40	0.21	0.54	0.13
m_5	0	0	0.66	0	0	0	0	0.34	0.26	0.58	0.14
m_6	0	0	0.84	0	0	0	0	0.16	0.49	0.69	0.18
m_7	0	0	0.92	0	0	0	0	0.08	0.67	0.74	0.20
m_8	0	0	0	0.24	0	0	0	0.76	0.00	0.49	0.01
m_9	0	0	0	0.78	0	0	0	0.22	2.00e-3	0.60	0.05
m_{10}	0	0	0	0.92	0	0	0	0.08	5.00e-3	0.66	0.06
m_{11}	0	0	0	0.99	0	0	0	0.01	0.03	0.68	0.06
m_{12}	0	0	0	0	0.12	0	0	0.88	0.12	0.51	0.05
m_{13}	0	0	0	0	0.57	0	0	0.43	0.57	0.65	0.22
m_{14}	0	0	0	0	0.51	0	0	0.49	0.50	0.61	0.20
m_{15}	0	0	0	0	0	0.08	0	0.92	0.06	0.54	0.02
m_{16}	0	0	0	0	0	0.57	0	0.43	0.51	0.60	0.12
m_{17}	0	0	0	0	0	0.80	0	0.20	0.76	0.68	0.17
m_{18}	0	0	0	0	0	0.99	0	0.01	0.99	0.75	0.21
m_{19}	0	0	0	0	0	0	0.47	0.53	0.14	0.54	0.02
m_{20}	0	0	0	0	0	0	0.76	0.24	0.37	0.63	0.03
$m_{\bigcirc\!\!\!\!\cap}$	0.99	8.00e-6	3.50e-5	0	0	0	0	0			
$m_{\oplus}$	0	0.18	0.81	0	4.00e-4	0	0	0			

rule or Dempster's rule. Let us discuss the results of each criterion individually :

- the degree of falsity is the sharpest criterion. It assigns large values to bbas in direct conflict with b like m_1 and m_{13} to m_{18}. However a large value c_7 (ranking 3) is also found for $m_7 = m_{\{b\}}^{0.08}$ because of its strong commitment. c_7 exceeds c_1 whereas m_7 supports b.

- In its raw form, $Conf_i$ is the criterion with the smallest variability and appears consequently less discriminative. The two bbas with the highest conflict value are $m_7 = m_{\{b\}}^{0.08}$ and $m_{18} = m_{\{a,c\}}^{0.01}$. As m_7 is in accordance with the majority opinion, this can be seen as a dangerous behaviour. Again, m_7 is considered more conflicting than m_1.

- Concerning criterion ξ_i, the smallest values are found for bbas m_8 to m_{12}, m_{19} and m_{20}. These bbas have a focal element of cardinal 2 that contains b. Like the two other criteria, m_7 is assigned a rather high value but it ranks 4th and we have $\xi_7 < \xi_1$.

More or less, the behaviour of each criteria can be justified on this example, but one cannot draw conclusions only on a single example. To allow a fair comparison, the following experiment is proposed: for each randomly generated set $\mathcal{S}$ of sbbas and for each criterion, the bba ranking first is removed until $\kappa_{s\subset\mathcal{S}} = 0$. These experiments were repeated 100 times and the average results are displayed in Table 3. The more efficient a criterion is, the less removals it

Table 3: Number of singular bba removals from set a $\mathcal{S}$ of randomly chosen sbbas until conflict is null. – means that the criterion could not be computed because the machine precision was reached.

$\lvert\mathcal{S}\rvert = M$	c_i	$Conf_i$	ξ_i
$M = 10$	3.80	6.46	3.86
$M = 20$	8.29	16.04	8.97
$M = 40$	17.91	35.63	19.89
$M = 80$	–	76.07	42.00

needs to obtain $\kappa_{s\subset\mathcal{S}} = 0$. To this regard, the sharpest criterion is the degree of falsity c_i followed by ξ_i whereas $Conf_i$ produces results significantly worse. It is important to stress that $Conf_i$ is the only criterion that is not defined based on the degree of conflict and that this may have an influence on its performances in this experiment. Moreover, if $m_1 = A^x$, $m_2 = A^{x'}$ with $x \neq x'$, we have $Conf_1 > 0$ and $\kappa_{\{m_1,m_2\}} = 0$. $Conf_i$ comprises other information than conflict-related one which may impair its capability to identify the bba to remove in priority. The major drawback of c_i remains its computational limitations when M is large. Looking at these conclusions, our criterion better distinguishes singular bbas than $Conf_i$ and is more robust than c_i. Note that ξ_i primarily removes sbbas with a singleton as a focal element as shown in Table 2. Consequently, after several removals, the remaining bbas may well be conflicting but without any pairwise conflict. q is thus unknown in this experiment.
Finally, one may wonder what are the computation times necessary for each of these approaches. Figure 4 shows that ξ_i computation is approximately twice that of c_i whereas $Conf_i$ is slower on small bba sets but faster on large bba sets. Note that in these experiments ξ_i is obtained in one loop (see algorithm 1).

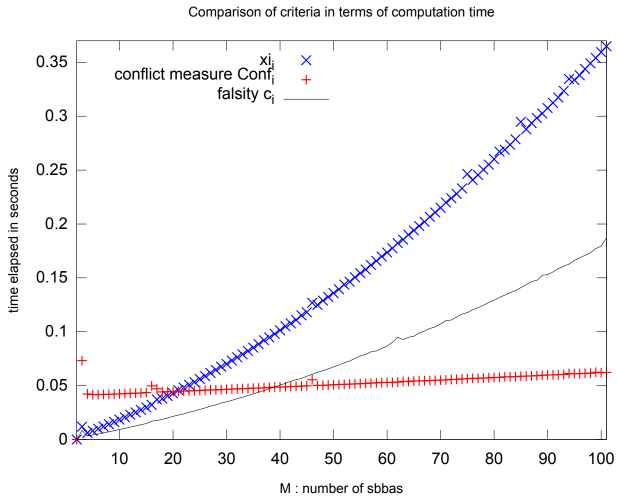

Figure 4: Computation time of criteria c_i, $Conf_i$ and ξ_i with respect to the number of bbas.

4.4. Sets of non-consonant bbas

The criteria discussed in this paper have been tested on sets of sbbas so far. There are two reasons accounting for this choice:

- Carrying experiments on general bbas often lead to subjective interpretations of the results because the information enclosed in a general bba is more difficult to interpret than that of a sbba.
- Conflict evaluation criteria are meant to be used on bbas that are direct output of sources of information in order to detect and post-process singular pieces of evidence. Most of the time, bba models [17, 18] produce sbbas so this type of bbas is the most widely used.

Nonetheless, it is interesting to observe the behaviour of the criteria when processing general bbas and to examine if their performances may degrade. Among non-simple bbas, we will only focus on a special kind of bbas that are non-consonant bbas. This type of bbas is the only one likely to provoke unforeseen issues. Indeed, a non-consonant bba is such that it has two focal elements A and B with $A \cap B = \emptyset$. In other words, there is some conflict encoded within the bba.

To understand the impact of a non-consonant bba on the criteria, let us carry this simple experiment: let us consider a set $\mathcal{S}$ of 20 sbbas. Suppose 5 of them are equal to $\{a\}^{0.5}$ and the 15 remaining one are equal to $\{b\}^{0.5}$. We already know how the three criteria respond to this situation. Now, let us process another bba set $\mathcal{S}'$ that contains 19 bbas. 4 of them are equal to $\{a\}^{0.5}$ and 14 of them are equal to $\{b\}^{0.5}$ and the 19th one is equal to $\{a\}^{0.5} \circledcirc \{b\}^{0.5}$.

Formally, the two sets contain the same pieces of evidence, but these pieces are

Table 4: Impact of a non-consonant bba on conflict analysis criteria

	Conflict analysis in $\mathcal{S}$				Conflict analysis in $\mathcal{S}'$					
	if $m_i = \{a\}^{0.5}$		if $m_i = \{b\}^{0.5}$		if $m_i = \{a\}^{0.5}$		if $m_i = \{b\}^{0.5}$		if $m_i = \{a\}^{0.5} \textcircled{\cap} \{b\}^{0.5}$	
	raw value	percent	raw value	percent	raw value	percent	raw value	percent	raw value	percent
ξ_i	7.49e-3	10.00%	2.50e-3	3.33%	7.65e-6	1.30%	2.51e-6	0.40%	5.35e-4	89.00%
$Conf_i$	0.28	7.30%	0.16	4.20%	0.28	7.90%	0.16	4.40%	0.25	8.00%
c_i	7.17-2	12.70%	1.37e-2	2.40%	7.17e-2	12.80%	1.37e-2	2.40%	8.16e-2	14.60%

not distributed the same way. The results for the three criteria are presented in Table 4.

In this experiment, it can be argued that each criteria produces a satisfactory response in its way. Indeed, for both the degree of falsity and the conflict measure, the raw values remain nearly unchanged and this can be regarded as normal since the same pieces of evidence are considered in both cases. In addition, the non-consonant bba is given a degree of falsity or conflict measure that is slightly higher than the singular bbas (*i.e.* when $m_i = \{a\}^{0.5}$). The criterion ξ_i has a dramatically different behaviour in the sense that the non-consonant bba drags 89% of the conflict. This is perfectly well understood when one looks at equation (29). The non-consonant bba is the only one that has a positive pairwise conflict with all other bba of $\mathcal{S}'$ and its value is a lot higher. This can be also regarded as an interesting result because non-consonant bba can be viewed as contradictory bba that deserves a to be processed in priority.

4.5. An example of practical use

In this subsection, we propose an example of practical use of conflict analysis criteria. This example is inspired from a weather forecast application presented in [19]. Suppose one has to choose between two hypotheses about tomorrow's weather: $\{rain\}$ or $\{sunshine\}$. This frame of discernment $\Omega_t = \{rain, sunshine\}$ is time-dependent and it is understood that the truth also evolves with time since the weather is not always rainy of sunny.

Now, let us further suppose that one has collected a sequence of bbas $\mathcal{S} = \{m_{t-w+1}, ..., m_t\}$ in order to make a decision on tomorrow's weather. The w past bbas are combined using the conjunctive rule with the present bba m_t. This approach works well when bbas belonging to the temporal window agree on the forecast but when it is not the case, the degree of conflict will rapidly be high valued, thus, tending to show that a safe decision is hard to be made.

To get a better picture of this phenomenon, let us examine the following simple case: one has collected w bbas that all equals to $\{rain\}^x$ followed by w that now equals to $\{sunshine\}^x$. Figure 5 is an illustration of this situation when $w = 3$. When the temporal window moves from past observations to new ones, the degree of conflict raises high and then decreases until the window comprises the w latest bbas that all support the $\{sunshine\}$ hypothesis. The degree of conflict appears to be insufficiently accurate in this situation as it cannot distinguish between these two cases:

- there is only one conflicting bba and this bba is the most recent one.

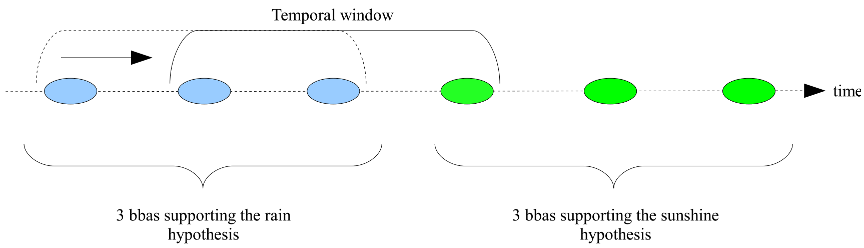

Figure 5: Example of a bba sequence with three bbas supporting the rain hypothesis followed by three more recent ones supporting the sunshine hypothesis with a temporal window size $w = 3$.

- there is only one conflicting bba and this bba is the oldest one.

In the first case the decision to be made is riskier than in the second case and consequently different processings are needed. This difficulty can be overcome by evaluating the contribution to the conflict of the most recent bba m_t. Figure 6 presents such evaluation using ξ_t, $Conf_t$ and c_t as well as the degree of conflict for two different window sizes and bba commitments. The values of ξ_t, $Conf_t$ and c_t are high when m_t brings some novelty and by thresholding these values it is possible to trigger and *ad hoc* post-processing. In this experiment, ξ_t is used with the following g function : $g(\xi_t) = \frac{\xi_t}{(1-x_{min})^2}$ with x_{min} the smallest x_i value among the set of w sbbas combined $\{m_i = A_i^{x_i}\}_{i=t-h}^{t}$. This allows $g(\xi_t)$ to be independent of the commitment if all sbbas are identically committed, *i.e.* $\forall i, x_i = x$ as in this experiment. Consequently, $g(\xi_t)$ is the only criterion that has the same value for the same proportion of singular bbas whatever w or x. It is thus easier to find a threshold for $g(\xi_t)$ that does not depend on w or on bba commitment.
Note that on real data, sbbas are very rarely identically committed. However, it is desirable to perfectly control the criterion behaviour in the identically committed case so as to analyse its values and extrapolate in the general case.

4.6. Concluding remarks on the experiments

Throughout this section, the proposed conflict analysis criterion ξ_i has been compared to two state-of-the-art approaches: the degree of falsity c_i and the conflict measure $Conf_i$. The experiments in subsections 4.1 and 4.2 have pointed out that ξ_i offers some possibilities beyond reach of the two other criteria. These possibilities are expressed via the homogeneity property and a situation where this property is useful is presented in subsection 4.5.
In terms of performances, it cannot be concluded that one of the criteria outperforms the other ones in all situations. Looking at the experiments in subsections 4.3, ξ_i should be preferred when the number of bbas is large but when it is

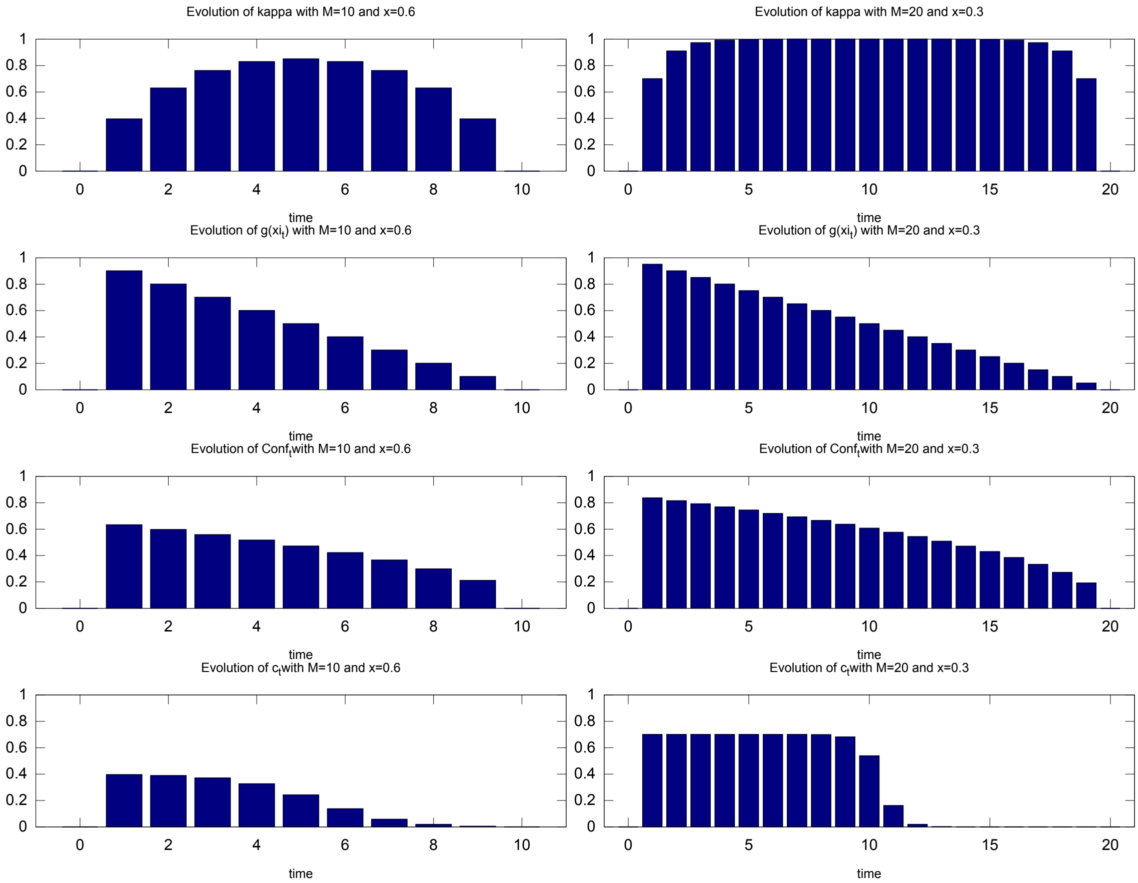

Figure 6: Evolution of the conflict evaluation criteria as the truth switches from a hypothesis to another one.

small, c_i obtains slightly better performances. $Conf_i$ appears to be less efficient when one intends to get rid of the most conflicting bbas but if another goal is sought, it may reveal itself as efficient as the two other ones. Moreover, ξ_i turns out to be useful when it is intended to mine non-consonant bbas in priority as shown in subsection 4.4.
The collection of the remarks and considerations above justify the practical interest of our contribution.

5. Conclusion

In this article, the way bbas conflict with one another has been studied under a new perspective. Viewing the degree of conflict as a function of discounting rates has led to new developments and the introduction of a new criterion assessing a bba contribution to the conflict. As compared to existing approaches, this criterion appears to be more robust to parameters like the proportion of conflicting bbas and the number of bbas with a better or equivalent efficiency. In addition the interpretation and the justification of this criterion are easily understood.

Various perspectives arise from this contribution on both theoretical and practical grounds. Concerning theoretical aspects, we would like to further investigate the relationship between sub-conflicts in connection with recent works on the discounting operation [12]. We would like also to investigate how this criterion could be re-injected inside a combination rule, Liu *et al.* [20] have recently proposed an approach following this idea but using another criterion.

We also intend to demonstrate the interest of this criterion through real-world problems. Detecting conflicting bbas allows to identify singular sources of information. These kind of sources correspond to deficient sources or to sources that collected a piece of information that cannot be perceived by the others. Concerning outlier detection, the criterion could be used to analyse datasets and mine data points of particular interest. Outlier detection is also essential in sensor networks. Some sensors may indeed be faulty or used in some conditions for which they are not calibrated, thus yielding unreliable information. Our criterion would be particularly useful in networks with a varying number of information sources.

Appendix A. Proofs of propositions

This appendix contains the proofs of the propositions presented in this article.

Appendix A.1. Proof of proposition 1

$\forall \mathcal{S} \subset \mathfrak{B}^{\Omega}$, $|\mathcal{S}| = M$, $\forall \vec{\alpha} \in [0,1]^M$, we have

$$\frac{\partial \kappa_{\mathcal{S}}}{\partial \alpha_i}(\vec{\alpha}) = \frac{\partial}{\partial \alpha_i}\left(\bigcirc\!\!\!\!\cap_{j=1}^{M} m_j^{\alpha_j}\right)(\emptyset) \tag{A.1}$$

Following the definition of the conjunctive rule, the expression becomes

$$\frac{\partial \kappa_{\mathcal{S}}}{\partial \alpha_i}(\vec{\alpha}) = \frac{\partial}{\partial \alpha_i} \sum_{\substack{\forall j, B_j \subseteq \Omega, \\ \cap_{j=1}^{M} B_j = \emptyset}} m_1^{\alpha_1}(B_1) \times \ldots \times m_M^{\alpha_M}(B_M) \tag{A.2}$$

Using the definition of the discounting operation on bba m_i, we get

$$\frac{\partial \kappa_{\mathcal{S}}}{\partial \alpha_i}(\vec{\alpha}) = \frac{\partial}{\partial \alpha_i}\left\{ \sum_{\substack{\forall j, B_j \subseteq \Omega, \\ \cap_{j=1}^{M} B_j = \emptyset \\ B_i \neq \Omega}} m_1^{\alpha_1}(B_1) \times ... \times (1-\alpha_i)\, m_i(B_i) \times ... \times m_M^{\alpha_M}(B_M) \right.$$

$$\left. + \sum_{\substack{\forall j, B_j \subseteq \Omega, \\ \cap_{j=1}^{M} B_j = \emptyset \\ B_i = \Omega}} m_1^{\alpha_1}(B_1) \times ... \times ((1-\alpha_i)\, m_i(\Omega) + \alpha_i) \times ... \times m_M^{\alpha_M}(B_M) \right\}$$

These sums can be re-organized as follows:

$$\frac{\partial \kappa_{\mathcal{S}}}{\partial \alpha_i}(\vec{\alpha}) = \frac{\partial}{\partial \alpha_i}\left\{ (1-\alpha_i) \sum_{\substack{\forall i, B_i \subseteq \Omega, \\ \cap_{i=j}^{M} B_j = \emptyset}} m_1^{\alpha_1}(B_1) \times ... \times m_i(B_i) \times ... \times m_M^{\alpha_M}(B_M) \right.$$

$$\left. + \alpha_i \sum_{\substack{\forall j \neq i, B_j \subseteq \Omega, \\ \cap_{j=1, j\neq i}^{M} B_j = \emptyset}} m_1^{\alpha_1}(B_1) \times ... \times m_M^{\alpha_M}(B_M) \right\}$$

$$= \frac{\partial}{\partial \alpha_i}\left\{ (1-\alpha_i)\, \kappa_{\mathcal{S}}(\vec{\alpha} - \alpha_i \vec{e}_i) + \alpha_i \kappa_{\mathcal{S}\setminus\{m_i\}}\left(p_{\mathcal{S}\setminus\{m_i\}}(\vec{\alpha})\right) \right\} \tag{A.3}$$

Since $\kappa_{\mathcal{S}}(\vec{\alpha} - \alpha_i \vec{e}_i)$ and $\kappa_{\mathcal{S}\setminus\{m_i\}}\left(p_{\mathcal{S}\setminus\{m_i\}}(\vec{\alpha})\right)$ are both independent from the variable α_i, using derivation rules, we obtain

$$\frac{\partial \kappa_{\mathcal{S}}}{\partial \alpha_i}(\vec{\alpha}) = -\kappa_{\mathcal{S}}(\vec{\alpha} - \alpha_i \vec{e}_i) + \kappa_{\mathcal{S}\setminus\{m_i\}}\left(p_{\mathcal{S}\setminus\{m_i\}}(\vec{\alpha})\right) \tag{A.4}$$

Appendix A.2. Proof of proposition 2

Proof by recurrence on the number M of bbas:

- Lets us first examine proposition 2 for $M = 2$: $\mathcal{S} = \{m_1, m_2\}$. In this case, the degree of conflict writes as:

$$\kappa_{\mathcal{S}}(\alpha_1,\alpha_2) = \sum_{\substack{B_1,B_2\subseteq\Omega,\\ B_1\cap B_2=\emptyset}} m_1^{\alpha_1}(B_1)\, m_2^{\alpha_2}(B_2) \tag{A.5}$$

Using the definition of the discounting operation on bba m_1, we get

$$\begin{aligned}\kappa_{\mathcal{S}}(\alpha_1,\alpha_2) &= \sum_{\substack{B_1,B_2\subseteq\Omega,\\ B_1\cap B_2=\emptyset\\ B_1\neq\Omega}} (1-\alpha_1)\, m_1(B_1)\, m_2^{\alpha_2}(B_2)\\ &\quad + \sum_{\substack{B_1,B_2\subseteq\Omega,\\ B_1\cap B_2=\emptyset\\ B_1=\Omega}} \left[(1-\alpha_1)\, m_1(\Omega)+\alpha_1\right] m_2^{\alpha_2}(B_2)\end{aligned}$$

These sums can be re-organized as follows:

$$\begin{aligned}\kappa_{\mathcal{S}}(\alpha_1,\alpha_2) &= (1-\alpha_1) \sum_{\substack{B_1,B_2\subseteq\Omega,\\ B_1\cap B_2=\emptyset}} m_1(B_1)\, m_2^{\alpha_2}(B_2)\\ &\quad +\alpha_1 \sum_{\substack{B_2\subseteq\Omega,\\ \Omega\cap B_2=\emptyset}} m_2^{\alpha_2}(B_2)\end{aligned}$$

The second sum reduces to a single term as the only subset B_2 of Ω such that $B_2\cap\Omega=\emptyset$ is $B_2=\emptyset$:

$$\begin{aligned}\kappa_{\mathcal{S}}(\alpha_1,\alpha_2) &= (1-\alpha_1) \sum_{\substack{B_1,B_2\subseteq\Omega,\\ B_1\cap B_2=\emptyset}} m_1(B_1)\, m_2^{\alpha_2}(B_2) + \alpha_1 m_2^{\alpha_2}(\emptyset)\\ &= (1-\alpha_1) \sum_{\substack{B_1,B_2\subseteq\Omega,\\ B_1\cap B_2=\emptyset}} m_1(B_1)\, m_2^{\alpha_2}(B_2) + \alpha_1(1-\alpha_2)\, m_2(\emptyset)\end{aligned}$$

For the remaining sum, one can apply the same process as above on bba m_2, and we thus have:

$$\kappa_{\mathcal{S}}(\alpha_1,\alpha_2) = (1-\alpha_1)\left[(1-\alpha_2) \sum_{\substack{B_1,B_2\subseteq\Omega,\\ B_1\cap B_2=\emptyset}} m_1(B_1)\, m_2(B_2) + \alpha_2 m_2(\emptyset)\right] + \alpha_1(1-\alpha_2)\, m_2(\emptyset$$

By definition of function κ_s, this expression writes as:

$$\kappa_{\mathcal{S}}(\alpha_1, \alpha_2) = (1-\alpha_1)(1-\alpha_2)\kappa_{\mathcal{S}} + (1-\alpha_1)\alpha_2\kappa_{\{m_1\}} + \alpha_1(1-\alpha_2)\kappa_{\{m_2\}}$$

Finally, following the definition of function f_s, we get

$$\kappa_{\mathcal{S}}(\alpha_1, \alpha_2) = f_{\mathcal{S}}(\alpha_1) f_{\mathcal{S}}(\alpha_2)\kappa_{\mathcal{S}} + f_{\{m_1\}}(\alpha_1) f_{\{m_1\}}(\alpha_2)\kappa_{\{m_1\}} + f_{\{m_2\}}(\alpha_1) f_{\{m_2\}}(\alpha_2)\kappa_{\{m_2\}}$$

The proposition is thus verified for $M = 2$.

- Let us now investigate proposition 2 at rank $M+1$ supposing that the it is verified at rank M (*i.e.* for any set s of bbas in $\mathcal{S} = \{m_1, ..., m_{M+1}\}$ such that $|s| = M$). We have

$$\begin{aligned}\kappa_{\mathcal{S}}(\vec{\alpha}) &= \left(\textcircled{\cap}_{j=1}^{M+1} m_j^{\alpha_j}\right)(\emptyset) \\ &= \sum_{\substack{\forall j, B_j \subseteq \Omega, \\ \cap_{j=1}^{M+1} B_j = \emptyset}} m_1^{\alpha_1}(B_1) \times ... \times m_{M+1}^{\alpha_{M+1}}(B_{M+1})\end{aligned} \tag{A.6}$$

with $\vec{\alpha} = (\alpha_1, ..., \alpha_{M+1})$. Using the same idea as in the case of $M = 2$, the expression can be expanded using the definition of the discounting operation on bba m_{M+1} :

$$\begin{aligned}\kappa_{\mathcal{S}}(\vec{\alpha}) &= \sum_{\substack{\forall j, B_j \subseteq \Omega, \\ \cap_{j=1}^{M+1} B_j = \emptyset \\ B_{M+1} \neq \Omega}} m_1^{\alpha_1}(B_1) \times ... \times (1-\alpha_{M+1}) m_{M+1}(B_{M+1}) \\ &+ \sum_{\substack{\forall j, B_j \subseteq \Omega, \\ \cap_{j=1}^{M+1} B_j = \emptyset \\ B_{M+1} = \Omega}} m_1^{\alpha_1}(B_1) \times ... \times [(1-\alpha_{M+1}) m_{M+1}(\Omega) + \alpha_{M+1}] \\ &= (1-\alpha_{M+1}) \sum_{\substack{\forall j, B_j \subseteq \Omega, \\ \cap_{j=1}^{M+1} B_j = \emptyset}} m_1^{\alpha_1}(B_1) \times ... \times m_{M+1}(B_{M+1}) \\ &+\alpha_{M+1} \sum_{\substack{\forall j, B_j \subseteq \Omega, \\ \cap_{j=1}^{M} B_j = \emptyset}} m_1^{\alpha_1}(B_1) \times ... \times m_M^{\alpha_M}(B_M) \\ &= (1-\alpha_{M+1})\kappa_{\mathcal{S}}(\alpha_1, ..., \alpha_M, 0) + \alpha_{M+1}\kappa_{\mathcal{S}\setminus\{m_{M+1}\}}(\alpha_1, ..., \alpha_M)\end{aligned} \tag{A.7}$$

In the expression above, the problem comes from the term $\kappa_{\mathcal{S}}(\alpha_1, ..., \alpha_M, 0)$ which remains a function of the $M+1$ variables. However, the same expansion can be applied to this term using the discounting operation definition for bba m_M and because the result obtained by equation A.7 is true $\forall\vec{\alpha}\in[0,1]^{M+1}$. We can thus write

$$\kappa_{\mathcal{S}}(\alpha_1,...,\alpha_M,0) = (1-\alpha_M)\,\kappa_{\mathcal{S}}(\alpha_1,...,\alpha_{M-1},0,0) \tag{A.8}$$
$$+\alpha_M\kappa_{\mathcal{S}\backslash\{m_M\}}(\alpha_1,...,\alpha_{M-1},0) \tag{A.9}$$

One may note that the term $\kappa_{\mathcal{S}\backslash\{m_M\}}(\alpha_1,...,\alpha_{M-1},0)$ is a function of the following M variables: $\{\alpha_1,...,\alpha_{M-1},\alpha_{M=1}\}$ (the value of $\alpha_{M=1}$ being 0 in this case).
Now expression A.9 can be inserted within expression A.7 and we have

$$\begin{aligned}\kappa_{\mathcal{S}}(\vec{\alpha}) &= (1-\alpha_{M+1})(1-\alpha_M)\,\kappa_{\mathcal{S}}(\alpha_1,...,\alpha_{M-1},0,0)\\ &\quad+(1-\alpha_{M+1})\,\alpha_M\kappa_{\mathcal{S}\backslash\{m_M\}}(\alpha_1,...,\alpha_{M-1},0)\\ &\quad+\alpha_{M+1}\kappa_{\mathcal{S}\backslash\{m_{M+1}\}}(\alpha_1,...,\alpha_M)\end{aligned} \tag{A.10}$$

The principle used to obtain expression A.10 can be iterated M times so as to obtain:

$$\begin{aligned}\kappa_{\mathcal{S}}(\vec{\alpha}) &= \prod_{i=1}^{M+1}(1-\alpha_i)\,\kappa_{\mathcal{S}}\left(\vec{0}\right)\\ &\quad+\alpha_1\prod_{i=2}^{M+1}(1-\alpha_i)\,\kappa_{\mathcal{S}\backslash\{m_1\}}(0,...,0)+...\\ &\quad+\alpha_j\prod_{i=j}^{M+1}(1-\alpha_i)\,\kappa_{\mathcal{S}\backslash\{m_j\}}(\alpha_1,...,\alpha_{j-1},0,...,0)+...\\ &\quad+\alpha_{M+1}\kappa_{\mathcal{S}\backslash\{m_{M+1}\}}(\alpha_1,...,\alpha_M)\end{aligned} \tag{A.11}$$

which can be written in a more condensed form as

$$\begin{aligned}\kappa_{\mathcal{S}}(\vec{\alpha}) &= \prod_{i=1}^{M+1}(1-\alpha_i)\,\kappa_{\mathcal{S}}\left(\vec{0}\right)\\ &\quad+\sum_{j=1}^{M+1}\alpha_j\prod_{i=j}^{M+1}(1-\alpha_i)\,\kappa_{\mathcal{S}\backslash\{m_j\}}(\alpha_1,...,\alpha_{j-1},0,...,0)\end{aligned} \tag{A.12}$$

Now we can use the hypothesis of recurrence at rank M for all the terms of the type $\kappa_{\mathcal{S}\backslash\{m_j\}}(\alpha_1,...,\alpha_{j-1},0,...,0)$. Indeed, we have

$$\kappa_{\mathcal{S}\backslash\{m_j\}}(\alpha_1,...,\alpha_{j-1},0,...,0) = \sum_{s\subseteq\mathcal{S}\backslash\{m_j\},s\neq\emptyset}\kappa_s\prod_{i=1}^{j-1}f_s(\alpha_i)\prod_{i=j+1}^{M+1}f_s(0) \tag{A.13}$$

Considering that $f_s(0)=\begin{cases}0 \text{ if } i\notin s\\ 1 \text{ if } i\in s\end{cases}$, all sets s that do not include $\{m_{j+1},...,m_{M+1}\}$ can be discarded. We thus have

$$\kappa_{\mathcal{S}\backslash\{m_j\}}(\alpha_1,...,\alpha_{j-1},0,...,0) = \sum_{s\subseteq\{m_1,...,m_{j-1}\},s\neq\emptyset}\kappa_{s\cup\{m_{j+1},...,m_{M+1}\}}\prod_{i=1}^{j-1}f_{s\cup\{m_{j+1},...,m_{M+1}\}}(\alpha_i) \tag{A.14}$$

Expression A.14 can be inserted in equation A.12:

$$\begin{aligned}
\kappa_{\mathcal{S}}\left(\vec{\alpha}\right) &= \prod_{i=1}^{M+1}\left(1-\alpha_i\right)\kappa_{\mathcal{S}}\left(\vec{0}\right) \\
&+ \sum_{j=1}^{M+1}\alpha_j\prod_{i=j}^{M+1}\left(1-\alpha_i\right)\sum_{s\subseteq\{m_1,\ldots,m_{j-1}\},s\neq\emptyset}\kappa_{s\cup\{m_{j+1},\ldots,m_{M+1}\}}\prod_{i=1}^{j-1}f_{s\cup\{m_{j+1},\ldots,m_{M+1}\}}\left(\alpha_i\right)
\end{aligned}$$

By defintion of function f_s, $f_{s\cup\{m_{j+1},\ldots,m_{M+1}\}}\left(\alpha_j\right)=\alpha_j$ because $m_j\notin s\cup\{m_{j+1},...,m_{M+1}\}$. Similarly, $\forall i>j$ $f_{s\cup\{m_{j+1},\ldots,m_{M+1}\}}\left(\alpha_i\right)=1-\alpha_i$, because $i\in s\cup\{m_{j+1},...,m_{M+1}\}$ and $\forall i$ $f_{\mathcal{S}}\left(\alpha_i\right)=1-\alpha_i$ because $m_i\in\mathcal{S}$. Consequently, we obtain

$$\begin{aligned}
\kappa_{\mathcal{S}}\left(\vec{\alpha}\right) &= \kappa_{\mathcal{S}}\prod_{i=1}^{M+1}f_{\mathcal{S}}\left(\alpha_i\right) \\
&+ \sum_{j=1}^{M+1}\sum_{s\subseteq\{m_1,\ldots,m_{j-1}\},s\neq\emptyset}\kappa_{s\cup\{m_{j+1},\ldots,m_{M+1}\}}\prod_{i=1}^{M+1}f_{s\cup\{m_{j+1},\ldots,m_{M+1}\}}\left(\alpha_i\right)
\end{aligned}$$

Let us now investigate the sets indexed as $s\cup\{m_{j+1},...,m_{M+1}\}$ in the double-sum in the expression above so as to determine if they can be re-organized. Let A be the set of all subsets of $\mathcal{S}$ except $\mathcal{S}$ itself:

$$A=\{s\subsetneq\mathcal{S}\}\,. \tag{A.15}$$

Let B the set that contains all sets of type $s\cup\{m_{j+1},...,m_{M+1}\}$ indexed in the double-sum:

$$B=\cup_{j=1}^{M+1}\{s\cup\{m_{j+1},...,m_{M+1}\}\text{ with }s\subseteq\{m_1,...,m_{j-1}\}\}\,. \tag{A.16}$$

Now let us compare A with B:

- A is the set of all subsets of $\mathcal{S}$ except $\mathcal{S}$ itself. B is composed of such subsets, therefore, clearly we have $B\subseteq A$.
- Looking at the double sum for $j=M+1$, we note that, we have $A_{M+1}=\{s\subseteq\{m_1,...,m_M\}\}\subseteq B$. This result is independent from the order with which the bbas are processed, so if one switches bba m_{M+1} with bba m_i using a permutation, the double sum remains identical and we have $A_i\subseteq B$. Consequently, $\forall i, A_i\subseteq B$, which implies $\cup_{i=1}^{M+1}A_i\subseteq B$ which by definition of A is equivalent to $A\subseteq B$
- Finally, since $B\subseteq A$ and $A\subseteq B$, we conclude $A=B$.

We thus obtain

$$
\begin{aligned}
\kappa_{\mathcal{S}}\left(\vec{\alpha}\right) &= \kappa_{\mathcal{S}} \prod_{i=1}^{M+1} f_{\mathcal{S}}\left(\alpha_i\right) \\
&\quad + \sum_{s \subsetneq \mathcal{S}, s \neq \emptyset} \kappa_s \prod_{i=1}^{M+1} f_s\left(\alpha_i\right) \\
&= \sum_{s \subseteq \mathcal{S}, s \neq \emptyset} \kappa_s \prod_{i=1}^{M+1} f_s\left(\alpha_i\right) \qquad \text{(A.17)}
\end{aligned}
$$

The hypothesis is thus verified at rank $M+1$ which concludes the proof of proposition 2.

Appendix A.3. Proof of proposition 4

Proof by recurrence on the number n of discounting operations with rates $\frac{1}{2}\vec{u}$:

- Lets us first examine proposition 4 for $n=1$: we have

$$
\begin{aligned}
\gamma_1^M\left(x\right) &= \frac{\left(2-1\right)^{M-x}}{2^M} \qquad \text{(A.18)} \\
&= \frac{1}{2^M}
\end{aligned}
$$

  This case reduces to corollary 1, it is thus verified.

- Let us now investigate proposition 4 at rank $n+1$ supposing that the it is verified at rank n. The result at rank 1 can be used on $\kappa_{\mathcal{S}}\left(\left[1-\left(\frac{1}{2}\right)^{n+1}\right]\vec{u}\right)$ to obtain

$$
\kappa_{\mathcal{S}}\left(\left[1-\left(\frac{1}{2}\right)^{n+1}\right]\vec{u}\right) = \sum_{s \subseteq \mathcal{S}, s \neq \emptyset} \gamma_1^M\left(|s|\right)\kappa_s\left(\left[1-\left(\frac{1}{2}\right)^{n}\right]\vec{u}\right).
$$

  Using the hypothesis at rank n, we can write

$$
\kappa_{\mathcal{S}}\left(\left[1-\left(\frac{1}{2}\right)^{n+1}\right]\vec{u}\right) = \sum_{s \subseteq \mathcal{S}, s \neq \emptyset} \gamma_1^M\left(|s|\right) \sum_{s' \subseteq s, s' \neq \emptyset} \gamma_n^M\left(|s'|\right)\kappa_{s'} \qquad \text{(A.19)}
$$

  The expression above is a weighed sum of sub-conflicts, it can thus be re-written as $\sum_{s \subseteq \mathcal{S}} \gamma_{n+1}^M\left(|s|\right)\kappa_s$ where $\gamma_{n+1}^M\left(|s|\right)$ are weights to determine. The values of weights can be found by analysing expression A.19. Let us separate this analysis for different values of $|s'|$ and count how many times each term $\kappa_{s'}$ occurs:

– when $|s'| = M$, there is only one possible choice: $s' = s = \mathcal{S}$. We conclude

$$\begin{aligned}
\gamma_{n+1}^{M}(M) &= \gamma_1^M(M)\,\gamma_n^M(M) \\
&= \frac{1}{2^M} \times \frac{1}{2^{nM}} \\
&= \frac{1}{2^{(n+1)M}}.
\end{aligned} \tag{A.20}$$

– when $|s'| = M - 1$, there are two possible choice for s. The term $\kappa_{s'}$ is found behind the coefficient:

  * $\gamma_1^M(M)\,\gamma_n^M(M-1)$ one time (case $s = \mathcal{S}$),
  * $\gamma_1^M(M-1)\,\gamma_n^{M-1}(M-1)$ one time (case $s = s'$).

  We conclude

$$\begin{aligned}
\gamma_{n+1}^{M}(M-1) &= \gamma_1^M(M)\,\gamma_n^M(M-1) + \gamma_1^M(M-1)\,\gamma_n^{M-1}(M-1) \\
&= \frac{1}{2^M} \times \frac{2^n - 1}{2^{nM}} + \frac{1}{2^M} \times \frac{1}{2^{n(M-1)}} \\
&= \frac{2^{n+1} - 1}{2^{(n+1)M}}.
\end{aligned} \tag{A.21}$$

– when $|s'| = M - i$ with $0 < i < M - 1$ (general case), the term $\kappa_{s'}$ is found C_i^j times behind the coefficient $\gamma_1^M(M-j)\,\gamma_n^{M-j}(M-i)$ with j such that $|s| = M - j$ and C_i^j the binomial coefficient (number of choices if j items among i). This statement is true for j varying from 0 to i. We can thus write

$$\begin{aligned}
\gamma_{n+1}^{M}(M-i) &= \sum_{j=0}^{i} C_i^j \gamma_1^M(M-j)\,\gamma_n^{M-j}(M-i) \\
&= \sum_{j=0}^{i} C_i^j \frac{1}{2^M} \frac{(2^n-1)^{M-j-(M-i)}}{2^{n(M-j)}} \\
&= \frac{1}{2^{M(n+1)}} \sum_{j=0}^{i} C_i^j (2^n-1)^{i-j} (2^n)^j \\
&= \frac{1}{2^{M(n+1)}} (2^n - 1 + 2^n)^i \\
&= \frac{\left(2^{n+1}-1\right)^i}{2^{M(n+1)}}.
\end{aligned} \tag{A.22}$$

The result above is equivalent to $\gamma_{n+1}^M(|s|) = \frac{\left(2^{n+1}-1\right)^{M-|s|}}{2^{M(n+1)}}$. Consequently, the hypothesis is verified at rank $n+1$, which concludes the proof of proposition 4.

Appendix B. Example of a bba set with global conflict but without pairwise conflict

Let $\Omega = \{a, b, c\}$ be a frame of discernment. Suppose one has obtained the following bba set: $\mathcal{S} = \{m_1, m_2, m_3\}$ with:

- $m_1 = \{a, b\}^x$
- $m_2 = \{a, c\}^x$
- $m_3 = \{b, c\}^x$
- $x \in]0, 1]$

We have $\kappa_{\mathcal{S}} > 0$ but $\kappa_{\{m_1,m_2\}} = \kappa_{\{m_1,m_3\}} = \kappa_{\{m_2,m_3\}} = 0$.

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